Solving Recurrence Relations Using Differential Operators

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Abstract

Linear recurrence relations are usually solved using the McLaurin series expansion of some known functions. That is, we assume that the coefficients a_n of $g(x) = \sum_{n=0}^{\infty} a_n x^n$ satisfy the recurrence relation. We then find an equation that involves $g(x)$ so that we may compare coefficients and get a closed form for a_n. We present another method using differential operators. In this case, we assume that the coefficients a_n in the series expansion of the function $h(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}$ satisfy the recurrence relation. The function $h(x)$ satisfies an initial value problem, which we solve using the methods of annihilators. The uniqueness of the solution allows us to find a closed form of a_n.

1 Introduction

We denote by $\mathbb{N}$ the set of natural numbers and by $\mathbb{R}$ the set of real numbers. The following definition was taken from [1].

Definition 1 *A k-order homogenous recurrence relation has the form* $a_{n+k} + b_1 a_{n+k-1} + b_2 a_{n+k-2} + ... + b_k a_n = 0$ *for some* $k, n \in \mathbb{N}$ *and* $b_1, b_2, ...\ b_k \in \mathbb{R}$ *and* $a_o, a_1, ..., a_{k-1}$ *have given values.*

The elements a_k, a_{k+1}, a_{k+2},....of the sequence$\{a_n\}$ that satisfy a recurrence relation can be computed successively. However, we wish to explore the possibility of finding a closed form expression for the n^{th} term a_n. A famous example of a second order homogenous recurrence relation is the Fibonacci sequence $a_{n+2} = a_{n+1} + a_n$, with $a_o = 1$ and $a_1 = 1$. In a recurrence relation, the higher terms in the relation is defined by the terms that come before it. Therefore

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in order to compute the 50^{th} term in the relation, it is sometimes necessary to know the first 49 terms. We would like to find an equation for a_n in terms of n so that we could calculate the 50^{th} term without having to know all the terms that come before it. The usual way that we solve a recurrence relation is to use the McLaurin series expansion of the function. We set

$$g(x) = \sum_{n=0}^{\infty} a_n x^n \tag{1}$$

where a_n satifies the reccurence relation. We then solve for $g(x)$, express it as a combination of well known functions, and if possible, find its McLaurin series expansion.

1.1 McLaurin Series Expansion Approach: An Example

Consider the recurrence relation

$$a_{n+2} - 2a_{n+1} + a_n = 0 \text{ where } a_0 = 3 \text{ and } a_1 = 2. \tag{2}$$

We multiply the relation by x^{n+2} and then sum it up from $n = 0$ to $n = \infty$ to get

$$\sum_{n=0}^{\infty} a_{n+2} x^{n+2} - 2\sum_{n=0}^{\infty} a_{n+1} x^{n+2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0. \tag{3}$$

We simplify the equation to get

$$\sum_{n=0}^{\infty} a_n x^n - 3 - 2x - 2x\sum_{n=0}^{\infty} a_n x^n + 6x + x^2\sum_{n=0}^{\infty} a_n x^n = 0. \tag{4}$$

Recall that $g(x) = \sum_{n=0}^{\infty} a_n x^n$ so that upon simplifying,

$$g(x) - 3 - 2x - 2xg(x) + 6x + x^2 g(x) = 0, \tag{5}$$

$$g(x) = \frac{-4x + 3}{(x-1)^2} = \frac{-4}{(x-1)} + \frac{-1}{(x-1)^2}. \tag{6}$$

Writing the McLaurin series expansion of the right hand side of equation (6)

$$g(x) = \sum_{n=0}^{\infty} 4x^n - \sum_{n=0}^{\infty} (n+1)x^n. \tag{7}$$

Thus $a_n = 4 - (n+1) = 3 - n$.

Now we can check by computing several of the values of a_n.

n	0	1	2	3	4
a_n	3	2	1	0	-1

2 Differential Operator Method

Another method to solve a recurrence relation is to use something other than the McLaurin series expansion. We set

$$h(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} \tag{8}$$

where a_n satifies the reccurence relation. To understand how this method works, we expand $h(x)$ and take successive derivatives. Notice that

$$h(x) = a_0 + \frac{a_1 x}{1!} + \frac{a_2 x^2}{2!} + \frac{a_3 x^3}{3!} + \frac{a_4 x^4}{4!} + \frac{a_5 x^5}{5!} + ... = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} \tag{9}$$

$$h'(x) = a_1 + \frac{a_2 x}{1!} + \frac{a_3 x^2}{2!} + \frac{a_4 x^3}{3!} + \frac{a_5 x^4}{4!} + ... = \sum_{n=0}^{\infty} \frac{a_{n+1} x^n}{n!} \tag{10}$$

$$h''(x) = a_2 + \frac{a_3 x}{1!} + \frac{a_4 x^2}{2!} + \frac{a_5 x^3}{3!} + ... = \sum_{n=0}^{\infty} \frac{a_{n+2} x^n}{n!}. \tag{11}$$

Using this information, we can solve the recurrence relation in a new way. Once we have the recurrence relation in terms of $h(x)$ and its derivatives, we can apply the method of differential operators (see page123 of [2]).

2.1 Differential Operator: Example 1

Consider the recurrence relation

$$a_{n+2} + 2a_{n+1} + a_n = 0 \text{ where } a_0 = 2 \text{ and } a_1 = 3. \tag{12}$$

Multiplying the relation by $\frac{x^n}{n!}$ and then summing it up from $n = 0$ to $n = \infty$,we get

$$\sum_{n=0}^{\infty} \frac{a_{n+2} x^n}{n!} + 2\sum_{n=0}^{\infty} \frac{a_{n+1} x^n}{n!} + \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} = 0. \tag{13}$$

Using equations (9), (10), and (11), we have

$$h''(x) + 2h'(x) + h(x) = 0. \tag{14}$$

Using D as the differential operator, we have

$$(D^2 + 2D + I)h(x) = 0. \tag{15}$$

This differential equation has a general solution

$$h(x) = Axe^{-x} + Be^{-x}. \tag{16}$$

Using the intial values $a_0 = h(0) = 2$ and $a_1 = h'(0) = 3$ we get $A = 5$ and $B = 2$, hence

$$h(x) = 5xe^{-x} + 2e^{-x}. \tag{17}$$

We now use the McLaurin series expansion of the right hand side of equation (14) to get

$$h(x) = \sum_{n=0}^{\infty} \frac{5n(-1)^{n+1}x^n}{n!} + \sum_{n=0}^{\infty} \frac{2(-x)^n}{n!}. \tag{18}$$

Thus $a_n = 5n(-1)^{n+1} + 2(-1)^n = (-1)^n(-5n + 2)$.

The following table shows several values of a_n.

n	0	1	2	3	4
a_n	2	3	-8	13	-18

2.2 Differential Operator: Example 2

Consider the reccurence relation

$$a_{n+2} - 2a_{n+1} + 2a_n = 0 \text{ where } a_0 = 1 \text{ and } a_1 = 0. \tag{19}$$

We begin by multipling the relation by $\frac{x^n}{n!}$ and then sum it up from $n = 0$ to $n = \infty$. Thus,

$$\sum_{n=0}^{\infty} \frac{a_{n+2}x^n}{n!} - 2\sum_{n=0}^{\infty} \frac{a_{n+1}x^n}{n!} + 2\sum_{n=0}^{\infty} \frac{a_n x^n}{n!} = 0. \tag{20}$$

By equations (9), (10), and (11)

$$h''(x) - 2h'(x) + 2h(x) = 0. \tag{21}$$

Hence,

$$(D^2 - 2D + 2I)h(x) = 0. \tag{22}$$

Completing the square, we get

$$((D - I)^2 + I)h(x) = 0. \tag{23}$$

A general solution of this differential equation is

$$h(x) = Ae^x \cos(x) + Be^x \sin(x). \tag{24}$$

Now using the initial conditions $a_0 = h(0) = 1$ and $\mathrm{a}_1 = h'(0) = 0$ we get $A = 1$ and $B = -1$, so

$$h(x) = e^x \cos(x) - e^x \sin(x). \tag{25}$$

Next we find the McLaurin series of the function $h(x)$.

2.3 Finding the McLaurin series of $e^x \cos(x) - e^x \sin(x)$

Using the earlier argument, we see that

$$h(x) = Ae^x \cos(x) + Be^x \sin(x) \tag{26}$$

is the general solution to the recurrence equation $a_{n+2} - 2a_{n+1} + 2a_n = 0$.

We would like to find the McLaurin series expansion of the function $h(x)$. In order to do this, we must have the values of a_n. The following can be easily verified.

Lemma 2 *Let $h(x) = Ae^x \cos(x)+Be^x \sin(x)$. Then $h'(x) = (A+B)e^x \cos(x)+ (B - A)e^x \sin(x)$.*

Notice that if $h(x)$ has the form $h(x) = Ae^x \cos(x)+Be^x \sin(x)$, then $h^{(k)}(x)$ has also the same form $h^{(k)}(x) = A_1e^x \cos(x) + B_1e^x \sin(x)$. Here, the values of A and B can be easily found if we know $h(0)$ and $h'(0)$: $A = h(0)$ and $B = h'(0) - A$.

Theorem 3 *If $h^{(k)}(0) = R$, then $h^{(k+4)}(0) = -4R$*

Proof. Suppose $h^{(k)}(0) = R$, and set $T = h^{(k+1)}(0) - R$. Then $h^{(k)}(x) = Re^x \cos(x) + Te^x \sin(x)$. Thus $a_k = R$, and using Lemma 2, we have $a_{k+1} = R + T$. Now, $a_{k+2} = 2a_{k+1} - 2a_k = 2T$, moreover we also have $a_{k+3} = 2a_{k+2} - 2a_{k+1} = 2T - 2R$. Finally, $a_{k+4} = -4R$. ■

For the specific case where $a_{n+2} - 2a_{n+1} + 2a_n = 0$, and $a_0 = 1$ and $a_1 = 0$, we have $a_{4k} = (-4)^k$ and $a_{4k+1} = 0$ for all positive integers k. Moreover, since $a_2 = -2$, then $a_{4k+2} = -2(-4)^k$. Finally, since $a_3 = -4$ we have $a_{4k+3} = (-4)^{k+1}$. This proves the following.

Theorem 4 *The recurrence relation $a_{n+2} - 2a_{n+1} + a_n = 0$ where $a_0 = 1$ and $a_1 = 0$, has solution $a_{4k} = (-4)^k$, $a_{4k+1} = 0$, $a_{4k+2} = -2(-4)^k$, and $a_{4k+3} = (-4)^{k+1}$, where k is a nonnegative integer.*

The following table shows the values of the first twenty elements of the sequence $\{a_n\}$.

$a_0 = 1$	$a_4 = -4$	$a_8 = 16$	$a_{12} = -64$	$a_{16} = 256$
$a_1 = 0$	$a_5 = 0$	$a_9 = 0$	$a_{13} = 0$	$a_{17} = 0$
$a_2 = -2$	$a_6 = 8$	$a_{10} = -32$	$a_{14} = 128$	$a_{18} = -512$
$a_3 = -4$	$a_7 = 16$	$a_{11} = -64$	$a_{15} = 256$	$a_{19} = -1024$

2.4 Differential Operator: Example 3

Consider the reccurence relations

$$a_{n+1} = (n + 1)a_n \text{ where } a_0 = 1. \tag{27}$$

We begin by multipling the relation by $\frac{x^n}{n!}$ and then sum it up from $n = 0$ to $n = \infty$. Thus,

$$\sum_{n=0}^{\infty} \frac{a_{n+1}x^n}{n!} = \sum_{n=0}^{\infty} \frac{(n+1)a_n x^n}{n!}. \tag{28}$$

We distribute the right hand side of the equation to get

$$\sum_{n=0}^{\infty} \frac{a_{n+1}x^n}{n!} = \sum_{n=0}^{\infty} \frac{n a_n x^n}{n!} + \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}. \tag{29}$$

By resetting our indices we can use equations (9), (10), and (11) to get

$$h'(x) = xh'(x) + h(x). \tag{30}$$

This is a separable differential equation:

$$\frac{h'(x)}{h(x)} = \frac{1}{1-x}. \tag{31}$$

Integrating, we obtain

$$\ln|h(x)| = -\ln|1-x| + C. \tag{32}$$

Solving for $h(x)$ we get

$$h(x) = \frac{\pm e^C}{1-x}. \tag{33}$$

Using the initial condition of $h(0) = 1$, we find

$$h(x) = \frac{1}{1-x}. \tag{34}$$

We now use the McLaurin series expansion of the right hand side of

$$h(x) = \sum_{n=0}^{\infty} x^n. \tag{35}$$

We need to get $h(x)$ in the correct form so we mulitply by $n!$

$$h(x) = \sum_{n=0}^{\infty} \frac{n!x^n}{n!}. \tag{36}$$

Now we can solve for our a_n to get

$$a_n = n!. \tag{37}$$

References

[1] K. P. Bogart. *Introductory Combinatorics, Second Edition.* Harcourt Brace Jovanovich, Orlando, 1990.

[2] C.H. Edwards and D. E. Penney. *Elementary Differential Equations Fifth Edition.* Pearson Prentice Hall, Upper Saddle River, 2004.