

Minimal Principles in Set Theory and Their Use in Ring Theory

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ABSTRACT

Minimal principles in Zermelo-Fraenkel set theory are discussed from an historical perspective. The Kuratowski-Moore minimal principle is used to establish the existence of minimal subobjects among certain types of prime ideals. The connection of these results to the Axiom of Choice is considered.

Minimal principles in set theory were considered as early as 1914, in Hausdorff's monumental textbook on set theory [3]. He reformulated his minimal principle in the 1927, 2nd ed. of this work. Unknown at that time to Hausdorff, Kuratowski had five years earlier established a similar minimal principle using the Axiom of Choice (A. C.), [9]. For the purpose of this note we will use a version of the minimal principle given by R. L. Moore in 1932, [13].

Minimal Principle: Let \mathcal{S} be a nonempty family of sets. If for each chain \mathcal{C} in (\mathcal{S}, \supseteq) , $\bigcap \mathcal{C}$ is in \mathcal{S} , then (\mathcal{S}, \supseteq) has a minimal element.

Moore deduced his minimal principle from the Well-Ordering Theorem, which is known to be equivalent to A. C. in Zermelo-Fraenkel (Z. F.) set theory. The Minimal Principle given here is easily seen to be equivalent to Moore's minimal principle, and that in turn is known to be equivalent to A. C. in Z. F. set theory. (For a discussion of minimal principles in set theory and their relation to A.C., see [12, p. 223, pp.231-232]).

Minimal principles which are equivalent to A. C. in Z. F. set theory have been used to establish important results in topology, by Kuratowski, R. L. Moore, and others. There are several propositions in ring theory where it seems natural to use one of these minimal principles in the proof, but instead one finds Zorn's Lemma arguments used, sometimes with complications that would be unnecessary had the Minimal Principle been used instead. This is particularly true in theorems which establish the existence of a minimal element among certain sets of prime ideals. In this note the Minimal Principle given above is used to prove one such result, and it is observed that this proof scheme can be readily used to prove similar results. The proof given seems more straight forward and natural than the usual ones found in the literature.

The concept of a prime ideal in an abstract commutative ring goes back to the work of Emmy Noether in the early 1920's, [14], where she put in an abstract setting ideas found earlier in work on algebraic number theory and algebraic geometry. In a commutative ring A an ideal I is a prime ideal if whenever $a, b \in A$ such that $ab \in I$, then $a \in I$ or $b \in I$. For rings in general this concept has been extended in several ways. The first is due to Krull in 1928, [8]. An ideal I in R is a prime ideal if when A and B are ideals of R such that $AB \subseteq I$, then $A \subseteq I$ or $B \subseteq I$. (Here R will always be a nonzero ring and "ring" means a not necessarily commutative ring, which may or may not have an identity element). McCoy, [10], showed this is equivalent to: if $a, b \in R$ such that $aRb \subseteq I$, then $a \in I$ or $b \in I$. When R is commutative the Krull generalization is equivalent to the earlier commutative ring version. The test of time has shown the Krull-McCoy concept is the most appropriate and fruitful version of "prime ideal" in general rings. However, the verbatim extension of the commutative, element-wise definition has also been investigated for rings in general, where it is called completely prime. (See [16], [11], and [1] for more on completely prime ideals).

The stage is now set to illustrate the use of the Minimal Principle.

Theorem 1 *Let I be an ideal of R and let Ω be the set of all completely prime ideals of R which contain I . Then (Ω, \supseteq) has a minimal element. Consequently, every ring R contains an ideal which is minimal in the set of all completely prime ideals R .*

Proof. Note that $R \in \Omega$; so $\Omega \neq \emptyset$. Let \mathcal{C} be a chain in (Ω, \supseteq) and let $T = \bigcap \mathcal{C}$. Then T is an ideal of R and $I \subseteq T$. If $a, b \in R$ such that $ab \in T$, then $ab \in X$, for each $X \in \mathcal{C}$. If $a \in T$, then there exists $X_0 \in \mathcal{C}$ such that $a \notin X$ for each $X \in \mathcal{C}$ with $X \subseteq X_0$. Thus, since each such X is completely prime, we have b is in each such X . Thus $b \in T$. Using $I = \{0\}$ yields the consequential result.

It is of interest to compare this proof in the case where R is commutative with the proofs using Zorn's Lemma which are standard in the literature, e.g., see [2, p. 84], [7, p. 6], [15, p. 53].

In a strictly analogous fashion one can use the Minimal Principle to prove the following result.

Theorem 2 *Let I be an ideal of R and let Λ be the set of all prime ideals of R which contain I . Then (Λ, \supseteq) has a minimal term. Consequently, every ring R contains a prime ideal which is minimal in the set of all prime ideals.*

A minimal prime ideal containing I is called the minimal prime belonging to I , [10]. McCoy established the existence of such minimal primes in a somewhat round about way using m -systems and Zorn's Lemma, [10, pp. 827-828]. The proof indicated for Theorem 2 is more direct, making no use of m -systems, and going directly to the desired minimal object via the Minimal Principle.

It is known [6] that in Z. F. set theory, the Axiom of Choice is equivalent to Krull's Theorem: Every commutative ring with unity has a maximal ideal. It was recently observed that the following statement is equivalent to A. C. in Z. F. set theory: a ring with a nonzero idempotent has a maximal right ideal, [4]. These results, taken together with the use here of the Minimal Principle, suggest the following open question.

Question. Is Theorem 1 (or Theorem 2) equivalent to A. C. in Z. F. set theory?

If the answer to this question is "yes," then it would be of interest to exhibit a ring without a minimal prime ideal in a model of Z. F. set theory without the Axiom of Choice. This would be in the spirit of Hodges examples of "Impossible rings", [5].

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