Generalizations of Primary Ideals

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ABSTRACT

This paper looks at a generalization of primary ideals in a noncommutative setting, gives some examples of this type of ideal, and examines some characteristics of the generalization.

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In a commutative ring, an ideal I is called primary if for any two ring elements a and b such that ab $\in$ I, then either a $\in$ I or $b^k \in$ I for some positive integer k. Primary ideals are a generalization of the concept of prime ideals. Much is known about primary ideals in commutative rings, but what sort of generalization might apply when we consider a ring which is not necessarily commutative? This is what we will consider in this paper. Some good sources for information on primary ideals in commutative rings, or on prime ideals are the references by Sharp [4] or Burton [1].

Throughout this paper we will let R represent a ring which is not necessarily commutative and not necessarily with unity (unless otherwise stated). Also we will use the notation $\mathbb{N}$ to represent the positive integers. We use the following definitions for a generalization of primary:

Definition 1 *:* An ideal I in a ring R is said to be *generalized right primary (generalized left primary)* if for any two ideals A, B in R such that if AB $\subseteq$ I, then either A $\subseteq$ I or $\exists k \in \mathbb{N}$ so that $B^k \subseteq$ I (B $\subseteq$ I or $\exists m \in \mathbb{N}$ so that $A^m \subseteq$ I). A ring is said to be *generalized right primary (generalized left primary)* if the zero ideal is generalized right primary (generalized left primary).

These definitions were given by Chatters and Hajarnavis [2] with slightly different terminology. Throughout the rest of the paper we use *g.r.p.* and *g.l.p.* to denote generalized right primary and generalized left primary respectively.

Note that a prime ideal is both *g.r.p.* and *g.l.p.* Also, in a commutative ring, *g.r.p* and *g.l.p.* each imply the ideal is primary. However, in general the two ideas are not equivalent. The following example demonstrates this:

Example 1 Let S be a semigroup with the elements $\{a, b\}$ and multiplication defined by the following table:

	a	b
a	*a*	*b*
b	*a*	*b*

Now let R be the semigroup ring $\mathbb{Z}_2[S]$. An easy calculation shows that this ring is *g.l.p.* but not *g.r.p.*

One of the first things one might want to know about *g.r.p.* ideals and *g.l.p.* ideals is where can we find examples of them. It is known that in commutative rings with unity, powers of maximal ideals are primary. See [1, p.83] for one proof of this. In fact we can show that in any ring with unity, powers of maximal ideals give us examples of *g.r.p.* and *g.l.p.* ideals.

Theorem 1 *Let R be a ring with unity, and M a maximal ideal of R. For $n \in \mathbb{N}$, M^n is a g.r.p. ideal and a g.l.p. ideal.*

Proof: For $n = 1$, M is a prime ideal and hence is *g.r.p.* and *g.l.p.* Take $n \geq 2$. Let A, B be ideals in R such that AB $\subseteq M^n$ and suppose A $\nsubseteq M^n$. Then $\overline{R}$ = R/M is a simple ring. Let $\overline{A}$ and $\overline{B}$ be the images of A and B in $\overline{R}$. We have four cases to consider:

(i) A $\subseteq$ M, B $\subseteq$ M;
(ii)A + M = R, B $\subseteq$ M;
(iii) A + M = R, B + M = R;
(iv) A $\subseteq$ M, B + M = R.

In cases (i) and (ii) B $\subseteq$ M implies B^n $\subseteq$ M^n. In case (iii) A + M = R implies AB + MB = RB. So B = RB = AB + MB $\subseteq$ M^n + M $\subseteq$ M. Thus B^n $\subseteq$ M^n.

Now assume case (iv) holds. Then A = AR = AB + AM $\subseteq$ M^n + M^2 $\subseteq$ M^2. But then A $\subseteq$ M^2 implies A = AR = AB + AM $\subseteq$ M^n + M^3 $\subseteq$ M^3. Continuing this process eventually gives A $\subseteq$ M^n, a contradiction.

Thus M^n is *g.r.p.* A similar proof shows M^n is *g.l.p.*

This yields a large class of ideals which are *g.r.p.* and *g.l.p.* We next consider how ideals which are *g.r.p.* (*g.l.p.*) can be used to find more such ideals.

Definition 2 For any ideal I of R let $\sqrt{I}$ be the sum of all the ideals of R for which some power of the ideal is contained in I. If A is any ideal for which some power is in I, we call A a *component* of $\sqrt{I}$.

Thus $\sqrt{I}$ is the preimage in R of all the nilpotent ideals in R/I. This is then one possible generalization to noncommutative rings of the classical radical of an ideal used in commutative ring theory.

Recall that any finite sum of nilpotent ideals is nilpotent. (For more on this see [3, p.19]). Thus any finite sum of components of $\sqrt{I}$ must be a component of $\sqrt{I}$. And since any $x \in \sqrt{I}$ is in some finite sum of components of $\sqrt{I}$, it follows that x is in a component of $\sqrt{I}$.

Lemma 1 *If A and B are ideals of a ring R then* $\sqrt{A \cap B} = \sqrt{A} \cap \sqrt{B}$.

Proof: If $x \in \sqrt{A}$ and $x \in \sqrt{B}$ then x is in a component of $\sqrt{A}$ and in a component of $\sqrt{B}$. Thus there exist ideals W and V of R and $m \in \mathbb{N}$ such that $x \in W \cap V$ and $W^m \subseteq A$, $V^m \subseteq B$. Consequently, $(W \cap V)^m \subseteq A \cap B$ and hence $x \in \sqrt{A \cap B}$. So $\sqrt{A} \cap \sqrt{B} \subseteq \sqrt{A \cap B}$. It is immediate from the definition of $\sqrt{\ }$ that $\sqrt{A \cap B} \subseteq \sqrt{A} \cap \sqrt{B}$.

Lemma 2 *Let* A_j*, j = 1,. . . , n be ideals of R and let* $A = \cap_{j=1}^{n} A_j$*. Then* $\sqrt{A} = \bigcap_{j=1}^{n} \sqrt{A_j}$.

Proof: The proof uses Lemma 1 and induction on n.

Lemma 3 *Let R be a ring in which every ideal is finitely generated, and let A, B be g.r.p. ideals in R. If* $\sqrt{A} = \sqrt{B}$*, then* $A \cap B$ *is a g.r.p. ideal of R.*

Proof: Let U, V be ideals of R such that $UV \subseteq A \cap B$, but $U \nsubseteq A \cap B$. This yields three possible cases:

(i) $\exists$ m, k $\in \mathbb{N}$ such that $V^k \subseteq A$ and $V^m \subseteq B$; then $V^q \subseteq A \cap B$, where $q = max\{m, k\}$.

(ii) $U \subseteq A$, $\exists m \in \mathbb{N}$ such that $V^m \subseteq B$;

(iii) $\exists k \in \mathbb{N}$ such that $V^k \subseteq A$, $U \subseteq B$.

Consider case (ii). V is finitely generated, say $V = (x_1, x_2, \ldots, x_n)$. Now for each i: $1 \leq i \leq n$, $x_i \in \sqrt{B} = \sqrt{A}$. Thus x_i is an element of a component W_i of $\sqrt{A}$.

Then $x_i \in W = \sum_{j=1}^{n} W_j$, for each i. Thus $V \subseteq W$. But W is a component of $\sqrt{A}$. So there exists $l \in \mathbb{N}$ such that $V^l \subseteq W^l \subseteq A$. And then $V^q \subseteq A \cap B$ where $q = max\{m, l\}$.

A similar proof holds for case (iii).

Finally, the we get the following result concerning intersections of *g.r.p.* ideals.

Theorem 2 *Let R be a ring in which every ideal is finitely generated, and let* A_j*,* $j = 1, \ldots, n$ *be g.r.p. ideals of R. If* $\sqrt{A_1} = \cdots = \sqrt{A_n}$ *then* $A = \bigcap_{j=1}^{n} A_j$ *is a g.r.p. ideal of R.*

Proof: The proof follows by induction on n, using Lemma 3.

References

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