

BÉZIER CURVES

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ABSTRACT. Over twenty-five years ago, Pierre Bézier, an automobile designer for Renault, introduced a new form of parametric curves that have come to be known as *Bézier curves*. These curves form an excellent source of examples for students who are studying parametric equations. These curves are especially interesting since they are easy to describe, easy to control, and can be formed into splines simply giving them multiple applications in computer aided geometric design, economics, data analysis, etc. We will define the basic structure of a Bézier curve, give some basic properties, discuss the use of these curves in splines, and give some problems appropriate for classroom use.

1. INTRODUCTION

The conic sections, the brachistochrone curve, cycloids, hypocycloids, epicycloids are all examples of very interesting curves that can be easily described and analyzed parametrically. Examples of each of these can be found in any introductory calculus book (e.g., [1], [3], etc.). The difficulties with these traditional examples of parametrization is a lack of practical applications associated with them. Too often for students, these examples seem to relate to a singular problem that has a well-known solution or describe a situation that is now fully understood. To better motivate the study of parametric equations students can consider the less well-known, but perhaps more practically important, example of the Bézier curve.

The idea underlying the Bézier curve lies in the weighting of the parametric functions by the coordinates of certain intermediate points. The intermediate points “attract and release the passing curve as if they had some gravitational influence,” [5]. We describe the Bézier cubic parametrically as

$$\begin{aligned}x(t) &= a_x(1-t)^3 + 3b_x(1-t)^2t + 3c_x(1-t)t^2 + d_x t^3, \\y(t) &= a_y(1-t)^3 + 3b_y(1-t)^2t + 3c_y(1-t)t^2 + d_y t^3, \\0 &\leq t \leq 1,\end{aligned}$$

where the coefficients are the coordinates of four **control points**: $A(a_x, a_y)$, $B(b_x, b_y)$, $C(c_x, c_y)$, and $D(d_x, d_y)$. The points A and D correspond to the initial and terminal points of the curve, respectively. The “intermediate points,” points B and C , determine the tangential direction at the initial and terminal points (see Fig. 1).

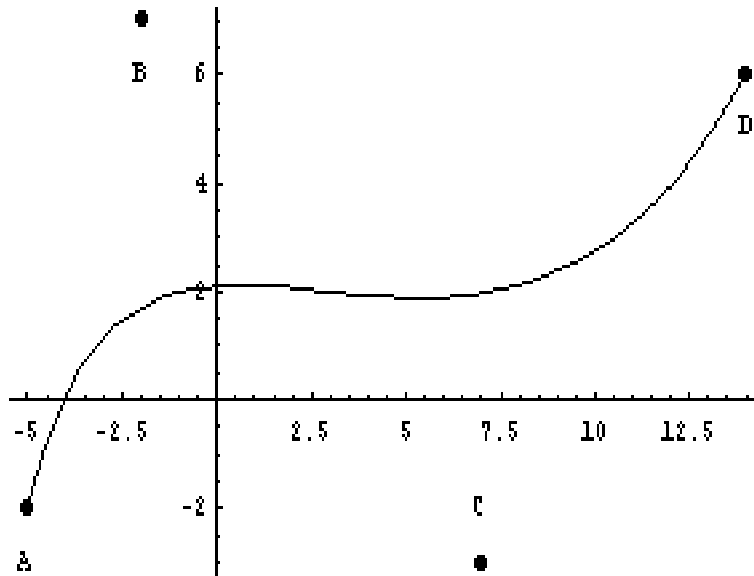


Fig. 1

A simple computation shows that

$$\frac{dy}{dx} \Big|_{(a_x, a_y)} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} \Big|_{t=0} = \frac{b_y - a_y}{b_x - a_x}.$$

Similarly,

$$\frac{dy}{dx} \Big|_{(d_x, d_y)} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} \Big|_{t=1} = \frac{d_y - c_y}{d_x - c_x}.$$

Thus, the tangent direction at A is determined by the vector \vec{AB} and the tangent direction at D is determined by the vector \vec{CD} . For a history of the development of Bézier curves and their applications in Computer Aided Design (CAD) and Computer Aided Machining (CAM) one should read the paper by Bézier, “The First Years of CAD/CAM and the UNISURF CAD System” in [4].

2. PROPERTIES

Since these curves are so easy to describe, they provide an entire family of new examples to consider when discussing parametric curves. It is very easy to graph these curves using a CAS or a graphing calculator. The use of such a graphing utility allows one to change the control points and easily view the resulting curve. Such experimentation may lead one to discover some interesting properties of Bézier curves.

The first such property derives from the fact that the points B and C can be changed without changing the tangent direction. How might this affect the graph? Consider the following two examples.

Take $A(0,0)$, $B(1,2)$, $C(6,3)$, and $D(4,0)$, Fig. 2.

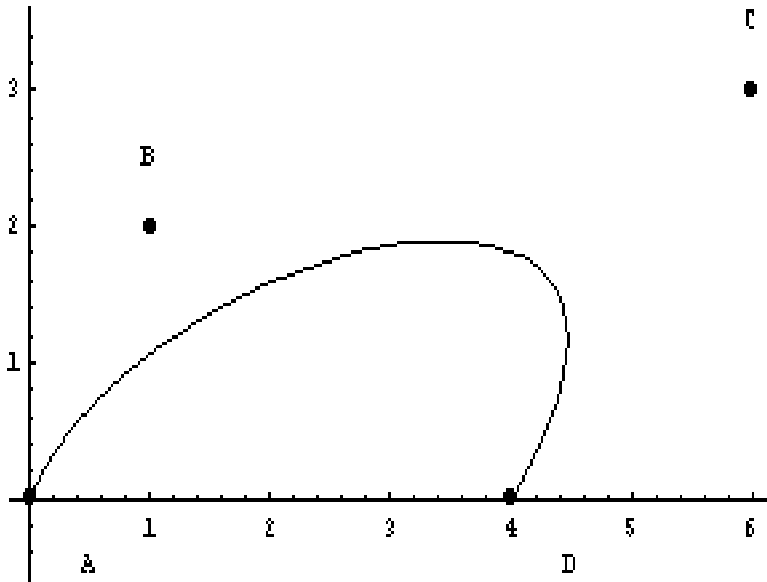


Fig. 2

Compare this to the Bézier cubic obtained when we take $A'(0,0)$, $B'(2,4)$, $C'(8,6)$, and $D'(4,0)$, Fig. 3.

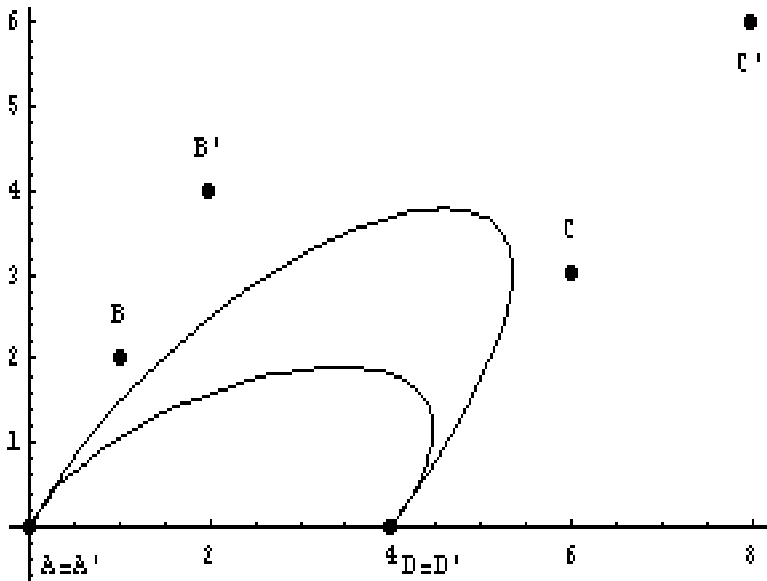


Fig. 3

We note that $\vec{B'A'} \parallel \vec{BA}$ and $\vec{D'C'} \parallel \vec{DC}$. Observe that when $|\vec{B'A'}| > |\vec{BA}|$ the curve “stays close to” the tangent line for a greater distance.

Second, the sharp eye may notice that in our two examples, the curves seem to be bounded by the convex polygon with vertices at the control points. In fact, this is a general result. To see why, we recall a general property of convex sets. If $P_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$, $1 \leq i \leq n$, are points in a convex body (say the convex hull of the control points), then every point $P = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ satisfying

$$P = \sum_{i=1}^n t_i P_i = \sum_{i=1}^n t_i \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

with $\sum_{i=1}^n t_i = 1$ is also in the convex body. Try this with $n = 2$ or 3 . Now any point P on a Bézier cubic satisfies

$$P = (1-t)^3 \begin{pmatrix} a_x \\ a_y \end{pmatrix} + 3(1-t)^2 t \begin{pmatrix} b_x \\ b_y \end{pmatrix} + 3(1-t)t^2 \begin{pmatrix} c_x \\ c_y \end{pmatrix} + t^3 \begin{pmatrix} d_x \\ d_y \end{pmatrix},$$

and since

$$(1-t)^3 + 3(1-t)^2 t + 3(1-t)t^2 + t^3 = ((1-t) + t)^3 = 1,$$

P is indeed in the convex hull of the control points. The fact that the curve has such a well-defined boundary gives Bézier curves a tremendous advantage over curves determined by some other interpolating techniques.

The final property that we should note is that these ideas can be generalized— we can consider the polynomials $x(t)$ and $y(t)$ with arbitrary order ($\deg_t x(t) = \deg_t y(t) = n = 2, 3, \text{dots}$), and we can also generalize to surfaces. In fact, both of these directions have been explored and exploited in practice, see [2], [4], or [6] for examples.

3. SPLINES

What makes Bézier curves more interesting than the traditional examples of parametric equations is the fact that these curves have many practical applications. For example, since both the position and tangential direction at the endpoints are determined in advance, it is very easy to form splines using Bézier curves.

A spline is an interpolating curve passing through a given set of points in the plane. Fitting a curve through specified points in the plane is an important tool in many fields including data analysis and design work. In many design problems one can use standard “curve-fitting” techniques. However, there is another class of design problems, the so called *ab initio* design problems, that require the designer to consider both aesthetic and functional criteria. In the *ab initio* design problem, curve-fitting techniques may not be effective. In these situations alternative techniques, such as the use of Bézier curves is a viable approach, ([4]). In the case of Bézier curves, for each point we can specify the direction of the tangent at that point, adjust the “intermediate points” accordingly, and create a separate Bézier curve passing through each consecutive pair of points. Consider the following example. We want to pass a smooth curve through the points $(0,0)$, $(4,0)$, and $(8,0)$ and having slope 2 at the point $(0,0)$, a slope of $\frac{3}{2}$ at the point $(4,0)$, and a slope of -1

at the point $(8,0)$. To achieve this, we will form a spline of two Bézier cubics. The first piece of the spline will be made using the points $\{(0,0), (0.5, 1), (6, 3), (4, 0)\}$ and the second piece will be made using the points $\{(4, 0), (3, -1.5), (7, -1), (8, 0)\}$. The two middle points in each set being the “intermediate points” for that piece. The resulting curve is shown in Fig. 4 below.

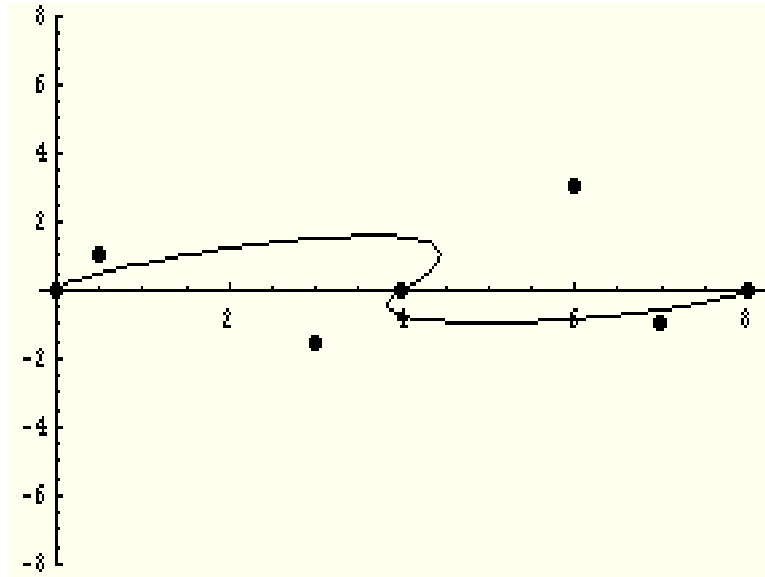


Fig. 4

We chose the “intermediate points” only under the restriction on the slopes at the three points on the curve. We could easily have chosen different “intermediate points” that would provide the same slopes, but a markedly different curve. If we had chosen the points determining the pieces of the spline to be $\{(0, 0), (4, 8), (8, 6), (4, 0)\}$ and $\{(4, 0), (0, -6), (0, -8), (8, 0)\}$ we would obtain the spline seen in Fig. 5.

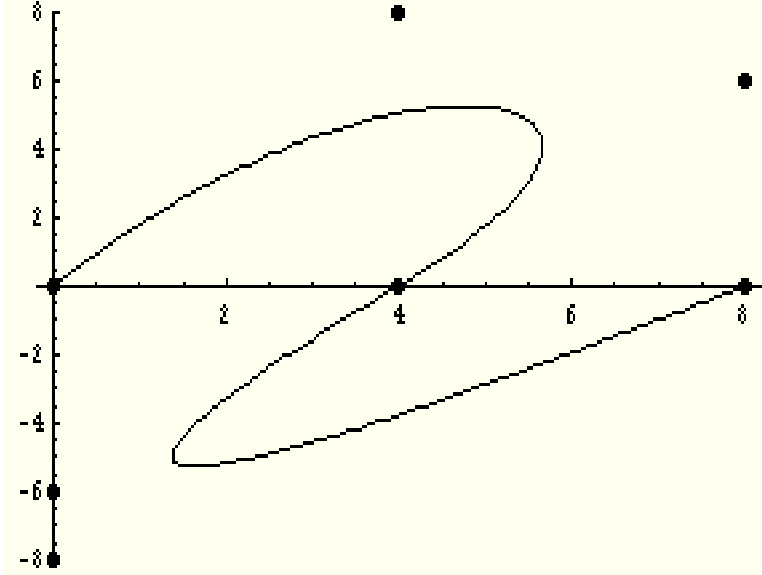


Fig. 5

We have described an informal approach to creating a spline having C^1 continuity at the joint. However, often one is concerned with creating continuity of various degrees at the joint between two splines, (see [4]). Suppose we have two Bézier curves $B(t)$ and $C(t)$, $t \in [0, 1]$, forming a spline. Let $\{P_0, P_1, \dots, P_n\}$ be the control points for the curve $B(t)$ and $\{Q_0, Q_1, \dots, Q_m\}$ be the control points for $C(t)$. The requirements for continuity are easily stated. If we want C^0 continuity then the only requirement is that $P_n = Q_0$. If C^1 continuity is desired then we need $B'(1) = C'(0)$. A brief computation (using $P_n = Q_0$) shows that this equation reduces to $Q_1 = \frac{m}{n}(P_n - P_{n-1}) + P_n$. Finally, a similar computation shows that C^2 continuity is obtained when $Q_2 = P_{n-1} - 4(P_{n-1} - P_n)$. In many applications, C^2 continuity is desirable.

Forming such splines allows one to create smooth curves with complex shapes. This is one reason why Bézier curves are so widely used in Computer Aided Geometric Design. (In fact, Pierre Bézier (1910-1999) developed these curves while working as an automobile designer for Renault.) These same attributes make these curves suitable for representing many of the typical curves found in economics including demand curves, supply curves, production curves, etc. Yet another application of Bézier curves is in the Adobe Postscript drawing model. Most obviously in the *curveto* command. This command takes four control points as arguments and returns the Bézier cubic determined by these points. If you have ever used *Adobe Illustrator*, *Macromedia Freehand*, or *Fontographer*, you have probably encountered Bézier curves.

4. PROBLEMS

The following questions are suitable for the undergraduate student.

1. Write the parametrization of the Bézier cubic with endpoints (2,3) and (5,5) and “intermediate points” of (1,4) and (6,10). Graph your curve.
2. Write the parametrization of of a Bézier cubic with endpoints of (0,0) and (4,4) and tangential slopes of 1/3 at (0,0) and of -2 at (4,4). Graph your curve. Compare your results with those of others.
3. Find a spline of Bézier cubics that approximates the unit circle. Estimate the maximum error between your spline and the unit circle.
4. The idea of a Bézier cubic can be extended to Bézier curves of any degree $n = 2, 3, 4, \dots$. How should one define a Bézier quadratic? A Bézier quartic? How many control points will be required? [Hint: Compare the expansion of $((1-t)+t)^3$ with the formula for the parametrization of the Bézier cubic discussed in this paper.]
5. Derive the conditions discussed in the previous section for C^1 and C^2 continuity.

5. CONCLUSION

We have seen that Bézier curves have a number of features that make them good examples to consider when exploring parameterization. These curves are easy to describe parametrically. They are easy to graph on a CAS or graphing calculator (the graphs in this paper were produced using *Mathematica*, but the author has reproduced each of them on a graphing calculator). The first derivative, $\frac{dy}{dx}$, is easy to compute and is in every case the ratio of quadratics (making it quite easy to determine where the curve has a vertical or horizontal tangent). Finally, Bézier curves have the advantage of being incredibly practical as evidenced by their use in Computer Aided Geometric Design and computer graphics for over twenty years.

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