

Paths for Bipeds

Margaret W. Maxfield

Professor, retired

Louisiana Tech University

mmaxf2@yahoo.com

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ABSTRACT

A biped traverses stepping-stones laid in a square, the right foot landing in a row or column parallel to that where the left foot lands. It is determined for routes between diagonal corners of the square what numbers s of steps acceptable paths can have, and how many paths there are for each s . Acceptable paths do not leave the square of stones and do not reuse any stones.

Paths for Bipeds

Let n^2 stepping-stones be arranged in a square, a stone at each (i,j) with $1 \leq i,j \leq n$. [Here we are using standard cartesian notation for lattice points in the plane.] A biped starts with left foot on $(1,1)$ and goes north and east to reach (n,n) . A similar path analysis would hold for beginning with the right foot on $(1,1)$, but with the transposed square (j,i) .

Three Requirements affect acceptability of paths:

- (1) The biped must face north or east [or, briefly, northeast], not south nor west.
- (2) The biped may not step outside the square. Each step must land on a stepping-stone.
- (3) Redundancy in a path is not acceptable; the biped may not return to a stepping-stone already used.

We define three double steps: N, E and D, assuming that each one starts with the left foot on some (i,j) , written $L(i,j)$.

N moves from $L(i,j)$ to $R(i+1,j+1)$ to $L(i,j+2)$. Although the second step goes west from $x = i+1$ to $x = i$, the biped is facing north during the entire double step, satisfying Requirement (1). The seeming contradiction is due to the bipedal motion in parallel columns $x = i$ and $x = i+1$.

E moves from $L(i,j)$ to $R(i+1,j-1)$ to $L(i+2,j)$. Here again, the first step from $y = j$ to $y = j-1$ seems to be a step south, but this is due solely to the fact that the right foot moves in the row $y = j-1$ parallel to the row $y = j$, where the left foot moves.

D moves from $L(i,j)$ to $R(i+1,j+1)$ to $L(i+2,j+2)$. The biped swivels on the right foot, effectively turning from north to east. The motion is along a northeast diagonal.

Restrictions for Double Steps are

1. No path may begin with double step E, since there is no stepping-stone at $(2,0)$. [Requirement 2]

2. No double step N may be followed immediately by double step E, to avoid redundancy: NE moves from $L(i,j)$ to $R(i+1,j+1)$ to $L(i,j+2)$ to $R(i+1,j+1)$ to $L(i+2,j+2)$. [Requirement 3]

3. If n is an odd number, then the last double step cannot be N, because to end with $L(n,n)$, N would have to go from $L(n,n-2)$ to $R(n+1,n-1)$ to $L(n,n)$, and there is no stone in column $n+1$. [Requirement 2] This restriction does not hold for even n .

The Minimum Path for each n is the path along the diagonal from $(1,1)$ to (n,n) . The number of steps s in the minimal path is $s = n-1$.

For odd $n = 2m+1$, the path code can be given as m D's. For instance, for $n = 9$, m is 4 and the path code for the shortest path is the diagonal D D D D, for which the number of steps $s = 2(4) = 9-1 = 8$.

For even $n = 2m$, the path code can be given as $(m-1)$ D's followed by a single step r of the right foot to (n,n) . For instance, for $n = 10$, m is 5 and the path code for the shortest path is the diagonal D D D D r , for which the number of steps s is $2(4) + 1 = 10-1 = 9$.

Paths of Length $s \geq n-1$ are constructed by including k N's and k E's in the path code. For $n = 2m+1$, each path code has $m+k$ double steps, comprising k N's, k E's and $m-k$ D's. The path requires $s = 2(m+k)$ single steps. For $n = 2m$, each path code has $m-1+k$ double steps, comprising k N's, k E's and $m-1-k$ D's, with the addition of a single step r at the end of the code. The path has $s = 2(m-1+k) + 1$ [single] steps.

In a 5×5 square of stones there are two paths: the shortest path D D has $k = 0$.

D $L(1,1)$ to $R(2,2)$ to $L(3,3)$, then D $L(3,3)$ to $R(4,4)$ to $L(5,5)$

The one longer path N D E has $k = 1$.

N $L(1,1)$ to $R(2,2)$ to $L(1,3)$, then D $L(1,3)$ to $R(2,4)$ to $L(3,5)$,
then E $L(3,5)$ to $R(4,4)$ to $L(5,5)$

In a 6×6 square the shortest path is D D r and there are two paths with $k = 1$, N D E r and D E N r . The paths D D r and N D E r are the same as those for $n = 5$ with the addition of the single step r at the end. The path D E N r ends in N for the double-step part, but does

not step outside the square of stones because of the single step r to come after the N . The detail of the path $D E N r$ is:

D L(1,1) to R(2,2) to L(3,3), then E L(3,3) to R(4,2) to L(5,3),
then N L(5,3) to R(6,4) to L(5,5), then r L(5,5) to R(6,6)

The number k of steps away from the diagonal can vary from 0 [for the shortest path] to $m-1$ for odd n or to $m-2$ for even n . For instance, when $n = 9$, $m = 4$, the maximum k is $m-1 = 3$. The only path with $k = 3$ for $n = 9$ has path code $N N N D E E E$, with $m+k = 7$ double steps, comprising 3 N's, 3 E's and $m-k = 1$ D. For $n = 10$, $m = 5$, the maximum k is $m-2 = 3$. There are "4-choose-3", or 4 paths with this maximum number of $s = n-1+2k = 2(m-1+k) + 1 = 15$ steps; their codes are

$D E E E N N N r$, $N D E E E N N r$, $N N D E E E N r$, $N N N D E E E r$

In these codes the N's are placed in positions 234, 134, 124 and 123 from the left, counting only N's and D's. The E's are placed from the right in positions 123, counting only E's and D's. The E's precede the N's to avoid redundancy, as explained above.

From the construction of path codes it can be seen that the number of paths for a given k is, for odd n , $\binom{m-1}{k}^2$ and for even n $\binom{m-1}{k} \binom{m-2}{k}$.

The Constructive Proof is completed by verifying that the code restrictions have been met.

For odd n , the last letter in the code must not be N . Why? Since there are k N's and $m-k$ D's in the path code, if the last letter were N it would be in position $(m-k)+k = m$, but the positions are all combinations of the first $m-1$ numbers, and do not include m .

Similarly, since the E's are placed from the right, a leading E would have position m , so that restriction is satisfied.

Turning to even n , there is no longer a restriction that N cannot be "last" in the double-step part of the code, because there is a single step r to follow, so N may be in any combination of k locations chosen from the first $m-1$ integers. The restriction that E cannot be last from the right [first from the right] is still in force, so only the first $m-2$ integers yield the combinations for placing E .

The Range of k can be inferred from the binomial coefficients, which according to convention yield zero if k is to be chosen from fewer than k positions. Thus for odd n we have $0 \leq k \leq m-1$ and for even n we have $0 \leq k \leq m-2$.

Are All Stones Reached by Paths? No, fewer than half the n^2 stepping-stones in the square are touched by a path of the sort studied here, though the ratio tends toward one-half as n increases. For $n = 2m+1$, paths touch $2m^2$ of the $(2m+1)^2$ stones. For $n = 2m$, paths touch $2m(m-1)$ of the $4m^2$ stones. Thus many stones in the square area could be removed, perhaps replaced by decorations, seating, and so on, without interrupting any of the path choices.

We have studied paths for just one biped. Could a second biped go from (n,n) to $(1,1)$ without bumping into the first one? What is the probability that paths constructed independently would result in collision, or would avoid collision?

The diagram shows all the paths at the same time for $n = 9$. The diagonal from (1,1) to (9,9) is shown horizontally. The path with $k = 2$ having code N D E E N D goes up over the box labeled 1A1 [A for “above the diagonal”], then down below the box labeled 1B2 [below the diagonal]. The path with $k = 2$ having code N D N D E E goes up over box 1A1, then up over box 2A2, then back down to the diagonal. Both these paths have $k = 2$ steps away from and back toward the diagonal, the first stepping both above and below the diagonal.

