

The Distance Between a Point and a Line Using Different Norms
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Abstract

We find the distance between a line and a point using three different norms and compare the results. Although there are cases when the distances are the same using different norms, in general, the distances are different. We have also noticed that the max-norm is less than or equal to the euclidean norm, which is less than or equal to the 1-norm.

1 Introduction

The length of a given vector is called a *norm* of that vector. Such length can also be thought of as a distance between two points in n -space. The most common norm is the euclidean norm given by

$$1.) \|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + |x_3|^2 + \dots + |x_n|^2}, \text{ where } x = [x_i] \in R^n.$$

However, there are several other norms.

Definition 1 A norm is a function $f : R^n \rightarrow R$ satisfying the following properties:

- a.) $f(x) \geq 0$ and $f(x) = 0$ iff $x = 0$
- b.) $f(cx) = |c| f(x)$
- c.) $f(x + y) \leq f(x) + f(y)$

The euclidean norm is a specific case of a family of norms called the p -norm, given by

$$\|x\|_p = \sqrt[p]{|x_1|^p + |x_2|^p + |x_3|^p + \dots + |x_n|^p}.$$

The euclidean norm is the case when $p = 2$. Another p -norm that we are concerned with is the 1-norm given by

$$2.) |x| = |x_1| + |x_2| + |x_3| + \dots + |x_n|$$

Here the absolute values of the dimensions are summed to calculate the distance. The 1-norm is used to calculate the distance in an alternate form of geometry, referred to as the *taxicab geometry*.

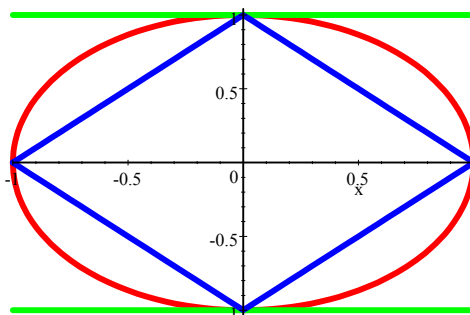
The max norm is an example of a norm that is not a member of the p -norm family. It is given by

$$3.) \|x\|_{\max} = \max\{|x_1|, |x_2|, |x_3|, \dots, |x_n|\}$$

With the euclidean norm, what is the use of any others? The other norms are useful in approximating the value of the euclidean norm. For example, to obtain an approximate value for the euclidean distance, one could use the max norm, which is easier to compute. The max norm is accurate when the absolute value

of one of the entries is far greater than the other entries. The 1-norm becomes more accurate at approximating the euclidean length than the max norm as the lengths of the dimensions become closer together. Generally the max norm underestimates the euclidean length and the 1-norm overestimates.

The differences in the norms are shown more clearly in the graph below. The points that are a distance 1 away from the origin are plotted for each norm. The euclidean norm is graphed in red and forms a circle about the origin; the max norm is shown in green and forms a square about the origin; and the 1-norm is in blue and creates a diamond. Notice that $\|x\|_{\max} \leq \|x\|_2 \leq |x|$.



2 Distance Between A Line And A Point

We want to calculate the distance between a line and a point. This distance using the euclidean norm was found by Michael Bard and Denny Himel [1]. What if we use a different norm? What effect, if any, will using different norms have on the outcome? Will the distance be the same? Will the point on the given line nearest to the given point be the same for all norms?

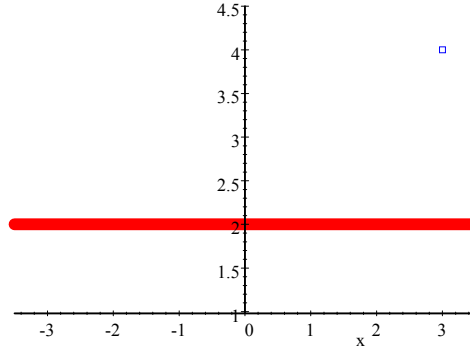
Given a line l , and a point p , to find the distance between l and p , we take a generic point q on l and find the distance between q and p . Then we choose the point q that is closest to p . This distance is the distance between p and l .

Although the following examples use specific lines and points the techniques used are applicable to any line and any point.

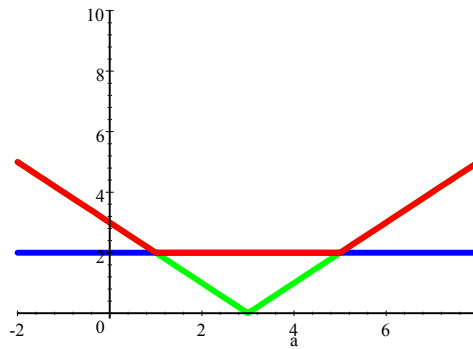
2.1 Example 1

Let us find the distance between the line $y = 2$ and the point $(3, 4)$.

Any point along the line $y = 2$ can be expressed as $(a, 2)$. The distance between the two points $(3, 4)$ and $(a, 2)$ can be expressed as the length of vector pointing from one point to another. This vector is $v = (3 - a, 2)$. Below is a graph of the line $y = 2$ and the point $(3, 4)$.

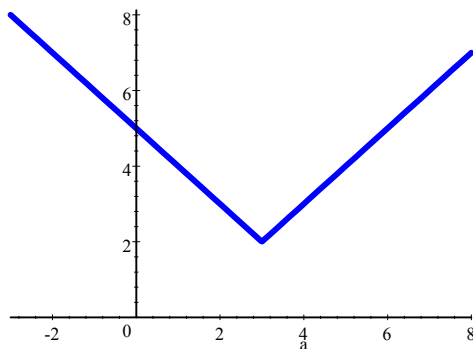


A) Using the $\|\cdot\|_{\max}$ norm. Remember that the max norm is $\|x\|_{\max} = \max\{|x_1|, |x_2|, |x_3|, \dots, |x_n|\}$. The expression for the length of vector v , using the max norm is $\|(3 - a, 2)\|_{\max} = \max\{|3 - a|, |2|\}$. The easiest way to determine the closest point is to graph the length as a function of a . The length function for a given value of a is the maximum of two functions $y = |3 - a|$ and $y = 2$. The three functions are graphed below; $y = |3 - a|$ is in green, $y = 2$ is in blue, and the maximum function is shown in red.



The length function hits a minimum of 2 over the interval $1 \leq a \leq 5$. This means that there is not a single closest point, but any point $(a, 2)$ such that $1 \leq a \leq 5$ is a closest point.

B) Using the $|\cdot|$ norm. Remember that $|(x, y)| = |x| + |y|$, so the length of the vector v using the 1-norm is given by $|(3 - a, 2)| = |3 - a| + |2|$. A graph is the simplest way to determine the minimum length in the 1-norm.



The minimum occurs at $a = 3$, where the component $|3 - a| = 0$, leaving only the absolute value of 2. Thus, the minimum distance is 2 and the point on the line closest to $(3, 4)$ is $(3, 2)$.

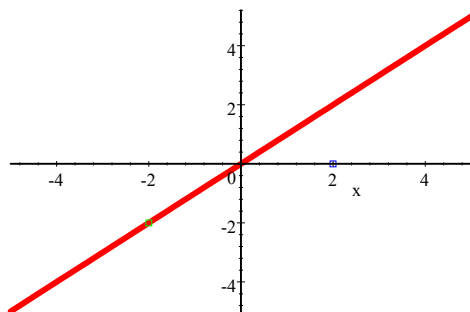
C) Using the $\|\cdot\|_2$ norm Remember, the euclidean norm is $\|(x, y)\|_2 =$

$\sqrt{|x|^2 + |y|^2}$. To simplify calculations, we look at $(\|(x, y)\|_2)^2 = |x|^2 + |y|^2$, that is, we look at $g(a) = (\|(3 - a, 2)\|_2)^2 = (3 - a)^2 + 4 = a^2 - 6a + 13$. To find the minimum distance we take the derivative of g respect to a and set it to zero: $g'(a) = 2a - 6 = 0$ This gives us $a = 3$. Hence, the closest point is $(3, 2)$, and the distance is 2.

In this example the distance between the line and the point was 2 for all three norms, and $(3, 2)$ was also always a closest point. Will the distance and the closest point always be the same for all norms?

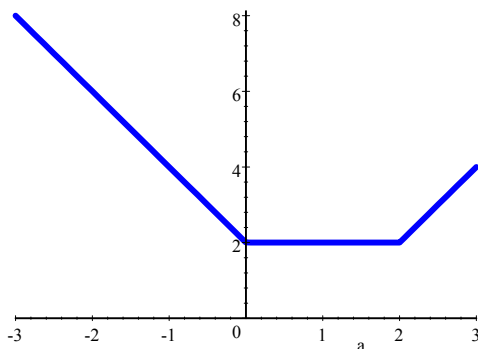
2.2 Example 2

Consider the line $y = x$ and the point $(2, 0)$. Notice that any point on the line $y = x$ can be written as (a, a) . The distance between $(2, 0)$ and (a, a) is the length of the vector $v = (a - 2, a)$. Below is a graph of the line $y = x$ and the point $(2, 0)$.



A) Using the $\|\cdot\|_2$ norm For the vector $v = (a - 2, a)$, we have $g(a) = (\|(a - 2, a)\|_2)^2 = 2a^2 - 4a + 4$. We set the derivative of $g(a)$ to zero: $g'(a) = 4a - 4 = 0$. Hence, $a = 1$, the closest point is $(1, 1)$, and the distance is $\sqrt{2 - 4 + 4} = \sqrt{2}$.

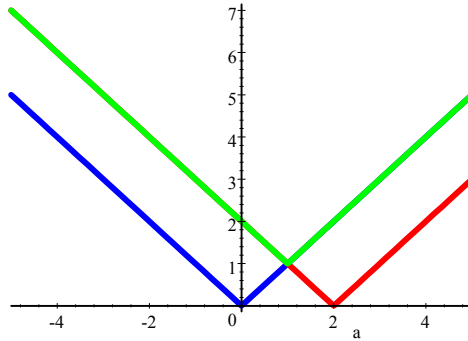
B) Using the $|\cdot|$ norm The length of vector $(a - 2, a)$, using the 1-norm, is $|(a - 2, a)| = |a - 2| + |a|$. If $a \leq 0$, then $|a| = -a$ and $|a - 2| = 2 - a$, so in this case, $y = 2 - 2a$. If $0 \leq a \leq 2$ then $|a| = a$ and $|a - 2| = 2 - a$, so that $y = 2$. If $a > 2$ then $|a| = a$ and $|a - 2| = a - 2$, so $y = 2a - 2$. These three line segments are graphed below.



The graph shows that the minimum distance is $|(a - 2, a)| = 2$ and it occurs when $0 \leq a \leq 2$.

C) Using the $\|\cdot\|_{\max}$ norm The most apparent way to determine the minimum length of the vector $(a - 2, a)$, using the maximum norm, is to graph the function, $\|(a - 2, a)\|_{\max} = \max\{|a - 2|, |a|\}$. This function can be tricky, so

it is easiest to graph the all functions within the max argument, and then trace over the maximum of the functions over the entire interval.



The blue line represents $y = |a|$ the red line represents $y = |a - 2|$ and the green line shows the maximum function. Looking at the graph, the minimum distance can be determined by knowing the general location where the red line intersects the blue line. In this case, the blue line is $d = 2 - a$, and the red line is $d = a$. Hence, $d = 2 - a = a$ at the point of intersection. It follows that $a = 1$ and $d = 1$, and the closest point on the line is $(1, 1)$. The value at which the lines intersect is the minimum distance. The minimum distance is $\|(a - 2, a)\|_{\max} = \max\{|a - 2|, |a|\} = \max\{|1|, |1|\} = 1$.

In this particular example, the distance between a line and a point was different for each norm: using the max norm, the distance was 1; using the 1-norm, the distance was 2; and using the euclidean norm, the distance was $\sqrt{2}$. Thus, the distance between a point and a line depends on the norm that is being used. Notice that there are infinitely many points on $y = x$ closest to $(2, 0)$ using the 1-norm, but there is a unique point closest to $(2, 0)$ using the max norm and the euclidean norm.

Bibliography

- [1] M. Bard and D. Himel. The Minimum Distance Between Two Lines in n Space. Proceedings of the LA-MS Section of the Mathematical Association of America. Spring 2001.