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Small Perimeter-Minimizing Double Bubbles in Compact Surfaces are Standard

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Abstract

We prove that in a smooth, compact, two-dimensional submanifold of R^N, the least-perimeter way to enclose and separate two regions of small prescribed areas is a standard double bubble, consisting of three constant-curvature curves meeting in threes at 120 degrees.

This paper is largely superseded by [MW], which proves that small stable double bubbles are standard.

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1. Introduction

Cotton and Freeman [CF, Conj. 1.1] conjectured that the least-perimeter way to enclose and separate two small prescribed volumes in a compact Riemannian manifold M is a standard double bubble (consisting of three constant-mean-curvature discs meeting along a curve at 120 degrees). Such a result was known only for R^2 [F], S^2 [Ma], H^2 [CF, Thm. 2.4], R^3 [HMRR], and R^4 [RHLS], where it was known for arbitrary prescribed volumes or areas. Our Theorem 2.1 provides a proof for any compact, two-dimensional submanifold of R^N.

1.1. The proof. The proof is a limit argument: in the small a surface is nearly Euclidean. The difficulty is the need for curvature bounds to obtain smooth convergence. If the ratio of the smaller area to the larger is bounded away from 0, such curvature bounds may be obtained from a rescaled limit bubble. If not, the rescaled limit is a circle, but the bubbles near the limit may have a large number m_i of tiny components of the smaller region. The requisite bound on m_i follows from a decomposition argument and an application of the isoperimetric inequality.

1.2. Higher dimensions. Generalizing the proof to n-dimensional manifolds M, even when the ratio of the smaller volume to the larger is bounded away from 0, would require knowing the result in R^n and knowing that convergence weakly and in measure, under bounded mean curvature, implies C^1 convergence, as is obvious for curves and known for (n-1)-dimensional surfaces without singularities [A, Sect. 8]. For $n > 2$, the argument needs a volume concentration lemma such as [M2, 13.7(1)] (with the partitioning of R^n into cubes replaced by a bounded-multiplicity covering of M by balls). Our approach does give an alternate proof of [MJ, Thm. 2.2] that small single bubbles are nearly round balls.

1.3. Existence and regularity. Geometric measure theory ([M4], cf. [M2, 13.4, 13.10]) proves that a perimeter-minimizing double bubble exists and consists of finitely many constant-curvature curves meeting in threes at 120 degrees. All curves separating the same two regions or exterior have the same curvature.

2. Small Double Bubbles are Standard

2.1. Theorem. Let M be a smooth, compact, two-dimensional submanifold of R^N. For small prescribed areas, a perimeter-minimizing double bubble is standard, i.e., consists of three constant-curvature curves meeting in two points at 120 degrees.

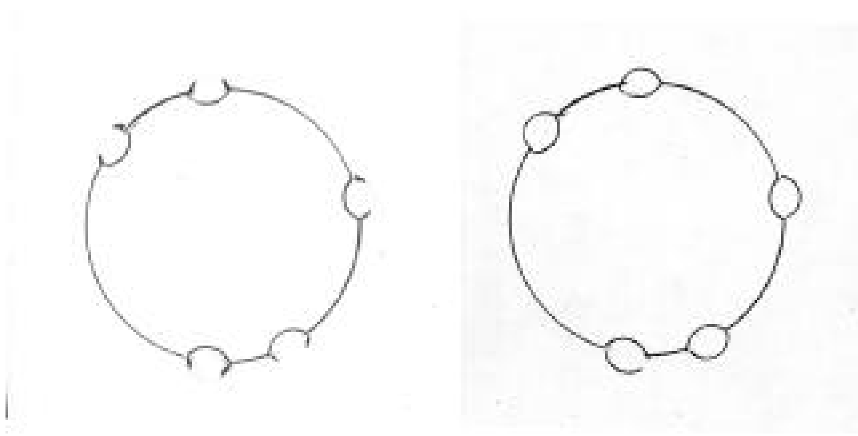
Proof. We claim that it suffices to show that a double bubble of small areas $A \geq A' > 0$, which minimizes perimeter among double bubbles of small areas at least A, A′, is standard. Indeed, given small areas A_0, A_0', minimize perimeter among double bubbles of small areas $A \geq A_0$, $A' \geq A_0'$. We may assume $A \geq A'$. The bubble certainly minimizes perimeter among double bubbles of small areas at least A, A′. By assumption it is standard, and therefore its curvatures κ_1, κ_2 are positive. Therefore it must have areas exactly A_0, A_0', or by reducing area one could reduce perimeter. Therefore a perimeter minimizer among double bubbles of areas A_0, A_0' is actually a perimeter minimizer among double bubbles of small areas at least A_0, A_0', proving the claim.

So consider a sequence of double bubbles B_i of small areas $A_i \geq A_i'$ > 0 converging to 0, which minimize perimeter among double bubbles of small areas at least A_i, A_i'. We may assume that $A_i'/A_i \to \lambda$. By the isoperimetric inequality, there is an $\varepsilon_1 > 0$ such that a connected component C_i of B_i maximizing the area of the first region has area at least $\varepsilon_1 A_i$ of the first region and that similarly a component has (maximal) area at least $\lambda\varepsilon_1 A_i$ of the second region. Let p_i, q_i be points of such components.

Move scalings of M by $A_i^{-1/2}$ tangent to the x_1,x_2-plane Π at the origin, with p_i at the origin. Henceforth we deal with such renormalizations. We may assume that C_i, together with any other components within unit distance of C_i, converge to a perimeter-minimizing bubble C in Π, with first region area at least ε_1, necessarily a standard double bubble, or circle if the second area vanishes. For some $\varepsilon_2 > 0$, there are smooth families of very localized deformations of R^N with $\Delta A' = 0$, $|\Delta P/\Delta A| < 2\varepsilon_1^{-1/2}$, for $|\Delta A| < \varepsilon_2$. (A circle in R^2 of area ε_1 has curvature $dP/dA = (\pi/\varepsilon_1)^{1/2} < 2\varepsilon_1^{-1/2}$. For a standard double bubble, the curvature is still smaller.)

Case 1: $\lambda > 0$. If $\lambda > 0$, one similarly obtains $\varepsilon_3 > 0$ and deformations with $\Delta A = 0$, $|\Delta P/\Delta A'| < 2(\lambda\varepsilon_1)^{-1/2}$, for $|\Delta A'| < \varepsilon_3$. Consequently, for i large, in the scaled bubbles B_i, one may make arbitrary small adjustments ΔA, $\Delta A'$ in area at cost at most $3\varepsilon_1^{-1/2}|\Delta A| + 3(\lambda\varepsilon_1)^{-1/2}|\Delta A'|$ (first by a limit argument with some $|\Delta A'| << |\Delta A|$ or some $|\Delta A| << |\Delta A'|$, and hence for arbitrary small $|\Delta A|$, $|\Delta A'|$ by linear combinations). There are two immediate consequences. First, the curvatures of the constant-curvature curves comprising the B_i are bounded by a single constant. Second, by the isoperimetric inequality, there are no very small components. We may assume that there are exactly m components. We may assume that each converges as above to a planar circle or standard double bubble. Minimization implies that there must be a single component converging (weakly) to the standard double bubble. Bounded curvature implies smooth convergence. Hence for large i, B_i is standard, the desired contradiction.

Case 2: $\lambda = 0$. If $\lambda = 0$, C must be a circle. Any other component must converge to 0, because a limit with more than one circle would not minimize perimeter. For i large, unless C_i is a nearly round circle, the main component of the first region in C_i is a curvilinear polygon with 2s ≥ 2 sides of alternating curvatures κ_1, $\kappa_0 = \kappa_2 - \kappa_1$ and 120-degree angles, as in Figure 1a, with $\kappa_0 > 0$. For the rest of C_i to fit together, C_i must be nearly a round component of the first region with small (nearly identical) bubbles of the second region, as in Figure 1b.



a. b.

Figure 1
The scaled limit C must be a nearly round circle with small bubbles of the second region.

We need a bound on the number m_i of little components of the second region. Since all are circles or digons of the same curvature, all have roughly the same perimeter and area, area at least $A_i'/2m_i$ and perimeter at least $\sqrt{A_i'/m_i}$ by the isoperimetric inequality. Let Q_i denote the least perimeter to enclose area A_i. Since the interface between the first region and the exterior, together with the shorter side of each digon, encloses area at least A_i, those perimeters contribute at least Q_i. Therefore the total perimeter is at least

$$Q_i + \Sigma \sqrt{A'/m_i} = Q_i + \sqrt{m_i A_i'/4} .$$

One way to enclose areas A_i, A_i' is with two near circles of perimeter less than $Q_i + \sqrt{4A_i'}$, as in Figure 2. Therefore $m_i A_i'/4 < 4A_i'$ and $m_i < 16$. Now we may assume that C_i is a nearly round circle with a fixed number m_1 of bubbles of the second region and that the rest of B_i is a fixed number m_2 of small nearly round bubbles of the second region. To remain minimizing in the limit, we must have $m_1 = 1$ and $m_2 = 0$, a standard double bubble, as desired.

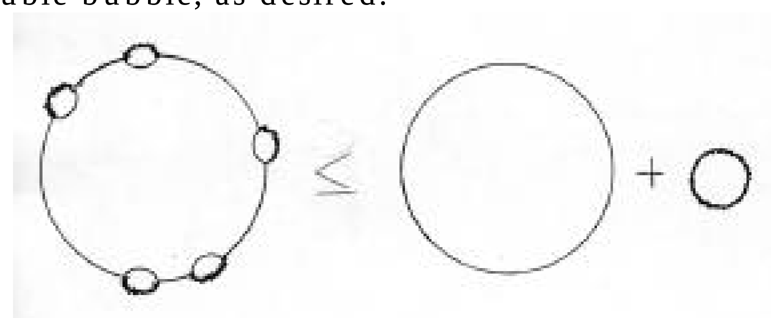


Figure 2
Since the double bubble must have less perimeter than two separate bubbles, it follows easily from the isoperimetric inequality that the small region has fewer than 16 components.

2.2. Immiscible fluids. As suggested in [CF, Sect. 2.1], Theorem 2.1 generalizes to the immiscible fluids problem, in which interfaces between different regions (or exterior) carry different costs (see [M2, Chapt. 16]).

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