

## Calculus Group I (1)

Find the equation of the line that passes through the point  $(1, 2)$  and cuts off the smallest area in the first quadrant.

## Calculus Group I (2)

Approximate the length of the longest ladder that can be taken from point  $A$  to point  $B$  in the figure below. Round your answer to the nearest hundredth of a foot. Assume that the ladder is dimensionless. (Hint: There is more than one way to take the ladder around the corner)

## Calculus Group I (3)

Prove that the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is  $\pi ab$ . There are a number of different ways to do this; use any method that works. You may assume that the area of a circle with radius 1 is known to be  $\pi$ .

## Calculus Group I (4)

Give an example of an indeterminate form of the type  $0^0$  which converges to  $+\infty$ , i.e, give examples of functions  $f(x)$  and  $g(x)$  such that:

i)  $\lim_{x \rightarrow +\infty} f(x) = 0$

ii)  $\lim_{x \rightarrow +\infty} g(x) = 0$

iii)  $\lim_{x \rightarrow +\infty} [f(x)]^{g(x)} = +\infty$

## Calculus Group I (5)

Find the sum of the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+1}{2^n n!}$$

Justify your answer. (Hint: Start with a Taylor series for  $e^{-x^2/2}$ )

## Abstract Algebra Group I (6)

Let  $G$  be a group. Define the function  $\phi : G \rightarrow G$  via  $\phi(x) = x^{-1}$ . Prove that  $\phi$  is an isomorphism if and only if  $G$  is Abelian.

## Advanced Calculus Group I (7)

Let  $I = [a, b]$  and let  $f : I \rightarrow \mathbb{R}$  be a continuous function such that  $f(x) > 0$  for all  $x \in I$ . Prove that there exists  $\alpha > 0$  such that  $f(x) \geq \alpha$  for all  $x \in I$ .

## Differential Equations Group I (8)

A body of mass  $m$  is in free fall in the earth's gravitational field. The only forces acting on the body are that due to gravity and air resistance. Assume that the acceleration due to gravity is  $9.8\text{m/sec}^2$ .

- a) Assuming that the force due to air resistance is proportional to the speed  $|v|$ , find the limiting or terminal velocity  $v_l$ . (Note that your answer will be in terms of the mass of the body and the constant of proportionality).
- b) Assuming that the body started from rest, under the same conditions as part (a) find the velocity  $v(t)$  as a function of time.
- c) Assuming that the force due to air resistance is proportional to a power  $|v|^r$  of the speed for some  $r > 0$ , find the limiting or terminal velocity  $v_l$ .

## Discrete Mathematics Group I (9)

- a) How many positive integers less than 1,000,000 are divisible by neither 3 nor 5?
- b) How many positive integers less than 1,000,000 are divisible by at least one of the numbers 3, 5 or 7?

## Linear Algebra Group I (10)

Let

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 \end{pmatrix}$$

Is  $A$  diagonalizable? Give explicit reasons for how you know.

## Abstract Algebra Group II (11)

Let  $G = \mathbb{R} \setminus \{-1\}$ , where  $\mathbb{R}$  stands for the set of real numbers. For each  $a, b \in G$  define  $a * b = a + b + ab$ .

- a ) Show that  $(G, *)$  is a group.
- b ) Compute  $(\frac{1}{2})^{-1}$  in  $(G, *)$ .

## Linear Algebra Group II (12)

Find all  $2 \times 2$  real matrices  $A$  so that

$$A \begin{pmatrix} -11 & -4 \\ 24 & 9 \end{pmatrix} A = \begin{pmatrix} -11 & -4 \\ 24 & 9 \end{pmatrix}$$

Carefully explain each step.

## Advanced Calculus Group II (13)

a) Define  $f(x) = kx^a$  where  $k, a > 0$ , then it follows that  $f(0) = 0$ ,  $f$  is continuous on  $[0, \infty)$ , and differentiable on  $(0, \infty)$ . Approximate  $k$  and  $a$  to four decimal places so that  $f$  satisfies the equation

$$f'(t) = \frac{1}{(f \circ f \circ f)(t)} \text{ for all } t > 0$$

b) Define  $f_n(x) = k_n x^{a_n}$  where  $k_n, a_n > 0$ , then it follows that  $f_n(0) = 0$ ,  $f_n$  is continuous on  $[0, \infty)$ , and differentiable on  $(0, \infty)$ . If  $f_n$  satisfies the equation

$$f'_n(t) = \frac{1}{\underbrace{(f_n \circ f_n \circ \cdots \circ f_n)(t)}_{n \text{ times}}} \text{ for all } t > 0$$

then find  $\lim_{n \rightarrow \infty} f_n(t)$ .

## Advanced Calculus Group II (14)

- a) Give an example of a Riemann integrable function  $f(x)$  defined on  $[0, 1]$  such that  $F(x) = \int_0^x f(t)dt$  is not differentiable at  $x_0 = \frac{1}{2}$ . Prove that your example works (5 points).
- b) Give an example of a Riemann integrable function  $f(x)$  defined on  $[0, 1]$  such that  $f(x)$  is not continuous at  $x_0 = \frac{1}{2}$  but  $F(x) = \int_0^x f(t)dt$  is differentiable at  $x_0 = \frac{1}{2}$ . Prove that your example works (5 points).

## Differential Equations Group II (15)

Let  $x(t)$  be a function of time  $t$  that satisfies the differential equation

$$x'(t) = (x - 1)(x - 2).$$

- a. (4 points) Show that if the initial value  $x(0)$  is less than 2, then  $x(t) \rightarrow 1$  as  $t \rightarrow \infty$ . What happens if  $x(0) = 2$ ? If  $x(0) > 2$ ?
- b. (6 points) Suppose that  $x(0) = 0$ . How large must the time  $t$  be in order that  $x(t) \geq 0.95$ ?

## Multivariable Calculus Group II (16)

In this problem  $\lfloor x \rfloor$  denotes the greatest integer less than or equal to  $x$ . Evaluate the double integrals.

a) 
$$\int_0^2 \int_0^2 \lfloor x + y \rfloor dx dy$$

b) 
$$\int_0^2 \int_0^2 \lfloor \sqrt{x^2 + y^2} \rfloor dx dy$$

## Numerical Analysis Group II (17)

Define

$$f(x) = \int_0^x \exp(-t^2) dt$$

and

$$g(x) = \exp(-x^2)$$

Show that there is exactly one positive solution to the equation  $f(x) = g(x)$  and approximate the solution to three decimal places.

## Topology Group II (18)

Let  $X = \mathbb{R}$ , the reals, and let  $T = \{U \subset X : 2 \in U \text{ or } U = \emptyset\}$ .

- Prove that  $T$  is a topology for  $X$  (4 points).
- Prove that the sequence  $\{\frac{1}{n}\}$  does not converge to 0 (3 points).
- Does  $\{\frac{1}{n}\}$  converge? To what? Prove it (3 points).

## Probability Group II (19)

A fair(?) coin is tossed 10,000 times, and the number of “heads” and “tails” are carefully recorded. Of the 10,000 tosses, 5200 landed heads-up, and 4800 landed tails-up. Do you believe that this is sufficient evidence to assert that the coin is biased? For whichever side you choose, write an argument (using complete sentences) to convince a skeptic, and include an adequate amount of mathematics to support your case.

## Set Theory Group II (20)

Let  $n \geq 1$  and define  $A = \{1, 2, \dots, n\}$ . Denote the power set of  $A$  by  $\mathbf{P}(A)$ . For each subset  $K \subset A$ , define the following functions:

- $s(K)$  = the sum of the members of  $K$ .
- $a(K)$  = the alternating sum of the members of  $K$ , starting with the largest element and continuing in decreasing order. For example,  $a(\{1, 4, 6, 7, 9\}) = 9 - 7 + 6 - 4 + 1$ . Find the following sums (justify your answers).

$$a) \sum_{K \in \mathbf{P}(A)} s(K)$$

$$b) \sum_{K \in \mathbf{P}(A)} a(K)$$